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AN ALGEBRAIC CHARACTERIZATION OF POWER SET IN COUNTABLE STANDARD MODELS OF ZF[1]

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§I. Introduction. The following is a classical result:

THEOREM 1.1. *A complete atomic Boolean algebra is isomorphic to a power set algebra* [2, *p.* 70].

One of the consequences of [3] is: If M is a countable standard model of ZF and $\mathfrak{A} \in M$ is a countable (in M) model of a complete $\aleph_0$-categorical theory T, then there is a countable standard model N of ZF and a $\Lambda \in N$ such that the Boolean algebra of definable (in T with parameters from $|\mathfrak{A}|$) subsets of $\mathfrak{A}$ is isomorphic to the power set algebra of Λ in N. In particular if $\mathfrak{A} = \langle \omega, \equiv \rangle$ and T is the theory of equality with additional axioms asserting the existence of at least n distinct elements for each $n < \omega$, then there is an N and $\Lambda \in N$ with $\langle P^N(\Lambda), \subseteq \rangle$ isomorphic to the countable, atomic, incomplete Boolean algebra of the finite and cofinite subsets of ω.

From the above we see that some incomplete Boolean algebras can be realized as power sets in standard models of ZF.

DEFINITION 1.1. A countable Boolean algebra $\langle B, \leq \rangle$ is a pseudo-power set if there is a countable standard model of ZF, N and a set $\Lambda \in N$ such that

$$\langle P^N(\Lambda), \subseteq \rangle \cong \langle B, \leq \rangle.$$

It is clear that a pseudo-power set is atomic. The problem seems to be to find the correct notion of pseudo completeness so that Theorem 1.1 generalizes. The need for such a notion is shown to be illusory by the following:

THEOREM 1.2. *Every countable atomic Boolean algebra is a pseudo-power set.*[2]

The proof will be presented in §III.

§II. Preliminary lemmas and constructions. Let M be a countable standard model of ZF. Let $\langle B, \leq \rangle \in M$ be an atomic Boolean algebra and let A be the set of atoms of B. In M we construct a ramified language $\mathscr{L}$ with limited quantifiers $\exists_\beta$, $\forall_\beta$, and with limited abstraction operators E^β, etc. $\mathscr{L}$ has two predicate symbols

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[2] After discovering Theorem 1.2, we found that B. Weglorz in 1968 had proved that every atomic Boolean algebra could be realized as a power set in a Fraenkel-Mostowski model of set theory with atoms. In [4] he reported that W. Marek had established a similar result for ZF. We do not know Marek's proof.

$\in$, $\equiv$; a constant $\mathbf{s}$ for each $s \in M$; constants $\mathbf{S}_{aj}$ for each $a \in A, j \in \omega$; constants $\mathbf{\Lambda}_a$ for each $a \in A$; constants $\mathbf{\Gamma}_b$ for each $b \in B$ (such that the sets

$$\{\langle \mathbf{S}_{aj}, a, j\rangle \mid a \in A, j \in \omega\}, \qquad \{\langle \mathbf{\Lambda}_a, a\rangle \mid a \in A\}, \qquad \{\langle \mathbf{\Gamma}_b, b\rangle \mid b \in B\}$$

are in M); and a constant $\mathbf{\Lambda}$.

Limited formulas and terms are defined as in [1]. A condition is a finite set of statements of the form $\mathbf{n} \in \mathbf{S}_{aj}$ or $\neg \mathbf{n} \in \mathbf{S}_{aj}$. We use $P, Q, Q', \ldots$ as variables for conditions. The empty condition is denoted by 0.

We define a weak forcing relation $\Vdash$ as follows:

$P \Vdash \mathbf{n} \in \mathbf{S}_{aj} \leftrightarrow \mathbf{n} \in \mathbf{S}_{aj}$ is in P.

$P \Vdash u \in \mathbf{\Lambda}_a \leftrightarrow \forall Q \supseteq P \exists Q' \supseteq Q \exists j \in \omega\ Q' \Vdash u \equiv \mathbf{S}_{aj}$.

$P \Vdash u \in \mathbf{\Lambda} \leftrightarrow \forall Q \supseteq P \exists Q' \supseteq Q \exists a \in A\ Q' \Vdash u \equiv \mathbf{\Lambda}_a$.

$P \Vdash u \in \mathbf{\Gamma}_b \leftrightarrow \forall Q \supseteq P \exists Q' \supseteq Q \exists a \in A\ a \leq b\ Q' \Vdash u \equiv \mathbf{\Lambda}_a$.

The remaining clauses in the definition of $\Vdash$ are the usual ones.

The definition of $\Vdash$ guarantees the following:

$$\forall a \in A, \forall j \in \omega (0 \Vdash \mathbf{S}_{aj} \in \mathbf{\Lambda}_a). \qquad \forall a \in A (0 \Vdash \mathbf{\Lambda}_a \in \mathbf{\Lambda}).$$
$$\forall b \in B, \forall a \in A (a \leq b \leftrightarrow 0 \Vdash \mathbf{\Lambda}_a \in \mathbf{\Gamma}_b).$$

By familiar arguments we obtain a complete sequence of conditions $\mathscr{P}$ and through $\mathscr{P}$ an interpretation Den of the terms of $\mathscr{L}$. This gives a transitive model N of ZF with $M \subseteq N$. (If necessary we write $N_{\mathscr{P}}$ to denote the dependence on $\mathscr{P}$.)

For terms $\tau \in \mathscr{L}$ we denote the corresponding interpretation in N by τ^N instead of Den(τ). We write S_{aj} for $\mathbf{S}_{aj}^N$, Λ_a for $\mathbf{\Lambda}_a^N$, Γ_b for $\mathbf{\Gamma}_b^N$, and Λ for $\mathbf{\Lambda}^N$. We have $S_{aj} \subseteq \omega$, $\forall a \in A, j \in \omega$, $\Lambda_a = \{S_{aj} \mid j \in \omega\}$, $\Lambda = \{\Lambda_a \mid a \in A\}$, $\Gamma_b = \{\Lambda_a \mid a \leq b\}$.

The following are important lemmas which we state without proof and which are quite familiar to the followers of forcing. Let Φ be any formula of $\mathscr{L}$.

LEMMA 2.1 (CONSISTENCY). *For no P do we have $P \Vdash \Phi$ and $P \Vdash \neg\Phi$.*

LEMMA 2.2 (EXTENSION). *If $P \Vdash \Phi$ and $P \subseteq Q$ then $Q \Vdash \Phi$.*

LEMMA 2.3 (TRUTH). *If $\mathscr{P}$ is a complete sequence then $N_{\mathscr{P}} \models \Phi \leftrightarrow \exists P \in \mathscr{P}\ P \Vdash \Phi$. And* $0 \Vdash \Phi \leftrightarrow \forall \mathscr{P} N_{\mathscr{P}} \models \Phi$.

Using these lemmas it is easy to prove $N \models \mathbf{S}_{aj} \equiv \mathbf{S}_{bk} \leftrightarrow a = b$ and $j = k$, $N \models \mathbf{\Lambda}_a \equiv \mathbf{\Lambda}_b \leftrightarrow a = b$, $N \models \mathbf{\Gamma}_a \equiv \mathbf{\Gamma}_b \leftrightarrow a = b$.

Let H be the group of permutations on ω which move only finitely many $j \in \omega$. We note that each element of H is in M.

For $\pi \in H, b \in B$ we define

$$\pi_b(\mathbf{S}_{aj}) = \mathbf{S}_{a\pi(j)} \quad \text{if } a = b,$$
$$= \mathbf{S}_{aj} \quad \text{otherwise.}$$

Then we define $\pi_b(P)$ and $\pi_b(\Phi)$ for conditions P and formulas Φ of $\mathscr{L}$ to be the condition or formula obtained by replacing each occurrence of $\mathbf{S}_{aj}$ by $\pi_b(\mathbf{S}_{aj})$.

LEMMA 2.4. $P \Vdash \Phi \leftrightarrow \pi_b(P) \Vdash \pi_b(\Phi)$, $\forall \pi \in H, b \in A$.

PROOF. By induction on the rank of Φ where rank is defined as in [1]. *Note.* rank($\mathbf{S}_{aj}$) $= \omega$, rank($\mathbf{\Lambda}_a$) $= \omega + 1$, rank($\mathbf{\Gamma}_b$) $=$ rank($\mathbf{\Lambda}$) $= \omega + 2$.

LEMMA 2.5 (CHOPPING DOWN). *If $P \Vdash \Phi$ and P_0 is the part of P mentioning only those $\mathbf{S}_{aj}$'s which appear in Φ, then $P_0 \Vdash \Phi$.*

PROOF. We write $P = P_0 \cup P_1$. Suppose not $P_0 \Vdash \Phi$. Then there is a $Q \supseteq P_0$, $Q \Vdash \neg\Phi$. We write $Q = P_0 \cup Q_1$. We can find π_b's whose product when applied to

P leave the $\mathbf{S}$'s in P_0 fixed and move the $\mathbf{S}$'s in P_1 in such a way that the resulting condition is consistent with Q_1. Call this product α. From Lemma 2.5 we have $\alpha(P) \Vdash \Phi$. By the choice of α, $\alpha(P) \cup Q$ is consistent. By Lemma 2.2, $\alpha(P) \cup Q \Vdash \Phi$ and $\alpha(P) \cup Q \Vdash \neg\Phi$. This contradicts Lemma 2.1. Hence $P_0 \Vdash \Phi$.

Let $\mathscr{L}(b_1, \ldots, b_n) \subseteq \mathscr{L}$ be the sublanguage whose terms and formulas mention at most $\mathbf{\Gamma}_{b_1}, \ldots, \mathbf{\Gamma}_{b_n}$. Let $\Phi \in \mathscr{L}(b_1, \ldots, b_n)$.

LEMMA 2.6. *$P \Vdash \exists\alpha\ x\ \ \Phi \leftrightarrow \forall Q \supseteq P \exists Q' \supseteq Q\ \exists u \in \mathscr{L}(b_1, \ldots, b_n)\ \operatorname{rank}(u) < \alpha$ $Q' \Vdash \Phi(u)$.*

PROOF. $\leftarrow$: Trivial by the definition of $\Vdash$ for bounded quantification.

$\rightarrow$: Let $Q \supseteq P$. Let $\mathscr{P}$ be a complete sequence containing Q. By Lemma 2.3, $N_{\mathscr{P}} \vDash \exists_\alpha x\ \Phi$. Here we think of $N_{\mathscr{P}}$ as an $\mathscr{L}(b_1, \ldots, b_n)$ structure. Hence there is a term $u \in \mathscr{L}(b_1, \ldots, b_n)$ $\operatorname{rank}(u) < \alpha$ with $N_{\mathscr{P}} \vDash \Phi(u)$. But then again by Lemma 2.3 there is $Q' \in \mathscr{P}$ (we may assume $Q' \supseteq Q$) such that $Q' \Vdash \Phi(u)$.

Let $c, d \in A$, $b_1, \ldots, b_n \in B$ be such that $c \leq b_i \leftrightarrow d \leq b_i$, $1 \leq i \leq n$. Define γ_d^c by

$$\gamma_d^c(\mathbf{S}_{aj}) = \begin{cases} \mathbf{S}_{dj} & \text{if } a = c, \\ \mathbf{S}_{aj} & \text{otherwise;} \end{cases} \qquad \gamma_d^c(\mathbf{\Lambda}_a) = \begin{cases} \mathbf{\Lambda}_d & \text{if } a = c, \\ \mathbf{\Lambda}_a & \text{otherwise;} \end{cases}$$
$$\gamma_d^c(\mathbf{\Gamma}_{b_i}) = \mathbf{\Gamma}_{b_i}, \qquad 1 \leq i \leq n.$$

We extend γ_d^c to conditions and formulas of $\mathscr{L}(b_1, \ldots, b_n)$ in the same manner that we extended π_b previously.

LEMMA 2.7. *For $\Phi \in \mathscr{L}(b_1, \ldots, b_n)$, $P \Vdash \Phi \leftrightarrow \gamma_d^c(P) \Vdash \gamma_d^c(\Phi)$.*

PROOF. By induction on the rank of Φ.

REMARKS. The restriction on c, d is needed to handle the $P \Vdash u \in \mathbf{\Gamma}_{b_i}$ case and Lemma 2.6 is needed for the bounded existential quantifier case.

§III. Main Theorem. We now prove

THEOREM 1.2. *Let M be a countable standard model of ZF, $\langle B, \leq\rangle \in M$ an atomic Boolean algebra. Then B is a pseudo-power set.*

PROOF. Let N be an extension of M as constructed in §II. Let $X \subseteq \Lambda$. We claim X is a Boolean combination of Γ_b's.

Let $\zeta(\mathbf{\Lambda}_{a_1}, \ldots, \mathbf{\Lambda}_{a_n}, \mathbf{\Gamma}_{b_1}, \ldots, \mathbf{\Gamma}_{b_m})$ be a term of $\mathscr{L}$ such that $\zeta^N = X$. We consider $\mathbf{\Lambda}_a$ to occur in ζ if some $\mathbf{S}_{aj}$ does.

Let $e \in \{0, 1\}$. Define

$$\Gamma_b^e = \Gamma_b \qquad \text{if } e = 1,$$
$$= \Lambda - \Gamma_b \quad \text{if } e = 0.$$

Let $C = \{\mathbf{\Lambda}_{a_1}, \ldots, \mathbf{\Lambda}_{a_n}\}$.

LEMMA 3.1. *Let $(e_1, \ldots, e_m) \in {}^m2$, let Λ_c, Λ_d be such that $\mathbf{\Lambda}_c, \mathbf{\Lambda}_d \notin C$. If $\Lambda_c \in \Gamma_{b_1}^{e_1} \cap \cdots \cap \Gamma_{b_m}^{e_m} \cap \zeta^N$ and $\Lambda_d \in \Gamma_{b_1}^{e_1} \cap \cdots \cap \Gamma_{b_m}^{e_m}$ then $\Lambda_d \in \Gamma_{b_1}^{e_1} \cap \cdots \cap \Gamma_{b_m}^{e_m} \cap \zeta^N$.*

PROOF. Let τ be the term

$$Ex^{\omega+1}\Big[\bigwedge_{i, e_1 = 1} x \in \mathbf{\Gamma}_{b_i} \wedge \bigwedge_{i, e_1 = 0} x \notin \mathbf{\Gamma}_{b_i}\Big], \qquad \tau^N = \Gamma_{b_1}^{e_1} \cap \cdots \cap \Gamma_{b_m}^{e_m}.$$

By Lemma 2.3 there is a P in the complete sequence defining N such that $P \Vdash \mathbf{\Lambda}_c \in \tau \wedge \mathbf{\Lambda}_c \in \zeta$. By Lemma 2.5 we may assume that P mentions only those

S_{a_j}'s which occur in τ and ζ. Consider γ_d^c. $\gamma_d^c(P) = P$, $\gamma_d^c(\tau) = \tau$; and since $\mathbf{\Lambda}_c$, $\mathbf{\Lambda}_d \notin C$, $\gamma_d^c(\zeta) = \zeta$. Thus, by Lemma 2.7,

$$\gamma_d^c(P) \Vdash \gamma_d^c(\mathbf{\Lambda}_c \in \tau \wedge \mathbf{\Lambda}_c \in \zeta)$$

and we have $P \Vdash \mathbf{\Lambda}_d \in \tau \wedge \mathbf{\Lambda}_d \in \zeta$. Since P is in the complete sequence, we have $\Lambda_d \in \tau^N \cap \zeta^N = \Gamma_{b_1}^{e_1} \cap \cdots \cap \Gamma_{b_m}^{e_m} \cap \zeta^N$.

COROLLARY 3.2. *Let* $\hat{C} = \{\Lambda_{a_i} \mid \mathbf{\Lambda}_{a_i} \in C\}$. *If* $(e_1, \ldots, e_m) \in {}^m2$ *and* $(\Gamma_{b_1}^{e_1} \cap \cdots \cap \Gamma_{b_m}^{e_m} - \hat{C}) \cap X \neq \varnothing$, *then* $(\Gamma_{b_1}^{e_1} \cap \cdots \cap \Gamma_{b_m}^{e_m} - \hat{C}) \subseteq X$.

REMARK. If $\Gamma_{b_1}^{e_1} \cap \cdots \cap \Gamma_{b_m}^{e_m} \cap X$ is infinite then $\Gamma_{b_1}^{e_1} \cap \cdots \cap \Gamma_{b_m}^{e_m} \subseteq X$ modulo a finite number of elements. This follows from the corollary since $\hat{C}$ is finite.

We now complete the proof of Theorem 1.2.

$$\Lambda = \bigcup\{\Gamma_{b_1}^{e_1} \cap \cdots \cap \Gamma_{b_m}^{e_m} \mid e \in {}^m2\}.$$

Thus

$$X = \bigcup\{\Gamma_{b_1}^{e_1} \cap \cdots \cap \Gamma_{b_m}^{e_m} \cap X \mid e \in {}^m2\}.$$

Let $I = \{e \in {}^m2 \mid \Gamma_{b_1}^{e_1} \cap \cdots \cap \Gamma_{b_m}^{e_m} \cap X$ is infinite$\}$. Then $X = \bigcup\{\Gamma_{b_1}^{e_1} \cap \cdots \cap \Gamma_{b_m}^{e_m} \mid e \in I\}$ modulo a finite number of elements. Now $\Lambda_a = \Gamma_a$ for $a \in A$. We can adjoin or delete the needed Λ_a's to $\bigcup\{\Gamma_{b_1}^{e_1} \cap \cdots \cap \Gamma_{b_m}^{e_m} \mid e \in I\}$ by Boolean operations in order to obtain X as a Boolean combination of Γ_b's.

Thus the (real world) correspondence $b \leftrightarrow \Gamma_b$ is an isomorphism between $\langle B, \leq \rangle$ and $\langle P^N(\Lambda), \subseteq \rangle$. And Theorem 1.2 is proven.

If we allow ourselves the luxury of Morse-Kelly set theory, we have

THEOREM 3.3 (MORSE-KELLY). *If B is a countable atomic Boolean algebra, then B is a pseudo-power set.*

PROOF. We can assume $\langle B, \leq \rangle \cong \langle \omega, \leq' \rangle$. Let M be a countable elementary submodel of the universe V such that $\langle \omega, \leq' \rangle \in M$. Now just apply Theorem 1.2.

Clearly every pseudo-power set is a countable atomic Boolean algebra.

COROLLARY 3.4 (MORSE-KELLY). *There is a* 1-1 *correspondence between countable atomic Boolean algebras and power sets in countable standard models of ZF.*

REMARKS. (i) There are $2^{\aleph_0}$ nonisomorphic countable atomic Boolean algebras which can be realized as pseudo-power sets.

(ii) Our methods say nothing about the nature of power sets in uncountable models of ZF.

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