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GEORGE METAKIDES

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§1. Introduction. Let α be a limit ordinal with the property that any "recursive" function whose domain is a proper initial segment of α has its range bounded by α. α is then called admissible (in a sense to be made precise later) and a recursion theory can be developed on it (α-recursion theory) by providing the generalized notions of α-recursively enumerable, α-recursive and α-finite. Takeuti [12] was the first to study recursive functions of ordinals, the subject owing its further development to Kripke [7], Platek [8], Kreisel [6], and Sacks [9].

Infinitary logic on the other hand (i.e., the study of languages which allow expressions of infinite length) was quite extensively studied by Scott [11], Tarski, Kreisel, Karp [5] and others. Kreisel suggested in the late '50's that these languages (even $\mathscr{L}_{\omega_1\omega}$ which allows countable expressions but only finite quantification) were too large and that one should only allow expressions which are, in some generalized sense, finite. This made the application of generalized recursion theory to the logic of infinitary languages appear natural. In 1967 Barwise [1] was the first to present a complete formalization of the restriction of $\mathscr{L}_{\omega_1\omega}$ to an admissible fragment $\mathscr{L}_A$ (A a countable admissible set) and to prove that completeness and compactness hold for it. [2] is an excellent reference for a detailed exposition of admissible languages.

In [4], Feferman shows the following:

1.1. THEOREM.[2] *To each recursively enumerable subset X of ω, there corresponds an axiomatizable theory $T(X)$ which has the same degree of unsolvability as X.*

In this paper the result of this theorem is generalized or "lifted" from ω to α where α is any admissible ordinal. The proof makes crucial use of an α-recursively enumerable theory for which elimination of quantifiers can be carried out in an "α-effective" way.

§2. Preliminaries. More detailed accounts of the fundamentals of α-recursion theory can be found in [6], [9] and [10].

2.1. DEFINITION. A nonempty transitive set A is *admissible* if A is closed under

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[2] This is an incomplete statement of Feferman's theorem which was given for arbitrary subsets X of ω.

transitive closure and the universal closures of the following formulas are true in A:

(a) Δ_0 *separation axiom.*

$$(\exists x)(\forall y)[y \in x \leftrightarrow y \in z \wedge \theta],$$

for all Δ_0 formulas θ with x not free in θ.

(b) Σ_1 *reflection principle.*

$$\theta \rightarrow (\exists y)[(y \text{ is transitive}) \wedge \theta^y],$$

for all Σ_1 formulas θ with y not free in θ.

2.2. DEFINITION. An ordinal α is *admissible* if L_α (the set of sets Gödel constructible by using ordinals less than α) is admissible.

Noting that an admissible L_α satisfies the Σ_1 replacement axiom of ZF, one obtains the basic definitions of recursion theory on an admissible ordinal α as follows:

2.3. DEFINITION. (i) A partial function f from α to α is *partial α-recursive* if its graph is Σ_1 over L_α.

(ii) A set X is *α-recursively enumerable* if it is the range of a partial α-recursive function.

(iii) A set X is *α-recursive* if both it and its α-complement are α-recursively enumerable.

(iv) A set X is *α-finite* if $X \in L_\alpha$ iff X is α-recursive and bounded by an ordinal less than α.

(v) A is *α-recursive in B* if there exist partial α-recursive functions ψ and φ such that, for all α-finite H,

$$H \subseteq A \leftrightarrow (\exists K)(\exists L)[\psi(K, L, H) = 0 \wedge K \subseteq B \wedge L \subseteq \bar{B}],$$
$$H \subseteq \bar{A} \leftrightarrow (\exists K)(\exists L)[\varphi(K, L, H) = 0 \wedge K \subseteq B \wedge L \subseteq \bar{B}],$$

where the quantifiers range over all α-finite sets. A and B are said to have the same *α-degree* if each is α-recursive in the other.

The *α-language* is obtained by starting with the ordinary atomic formulas and closing under negation, α-finite disjunction and conjunction and finite quantification. The logical axioms and rules of inference are the usual ones with the obvious generalizations that make them apply to formulas of α-finite length.

§3. The elimination of quantifiers. In what follows, α is an arbitrary admissible ordinal.

The device used by Feferman in his proof of 1.1 amounts to coding the r.e. set X as negations of assertions about the number of distinct elements in the domain. If T is the theory based on identity, then

$$T(X) = T \cup \{\neg(\text{there exist exactly } n \text{ elements in the domain}) \mid n \in X\}$$

closed under deduction. The same coding can be used again to obtain a theory $T(X)$ with an α-r.e. set of axioms and the same α-degree as any given α-r.e. subset X of ω. The proof of this proceeds by an elimination of quantifiers and the only difficulty is due to the fact that the general distributive law used in most classical eliminations of quantifiers fails to hold for α-logic. This difficulty is overcome thanks to the Lindenbaum α-algebra[3] of statements being an easily understood atomic Boolean algebra. Another proof of a stronger result in the ordinary case ($\alpha = \omega$) uses a theory whose Lindenbaum algebra of statements has an atomless part with a recursive free basis $P_1, P_2, \cdots$ and the coding $T(X)$ = theory of $\{\neg P_i \mid i \in X\}$. A direct lifting of this proof to α-logic fails, for the large fragment of propositional α-logic generated by $P_1, P_2, \cdots$ already has a set of tautologies which is only α-r.e. and not α-recursive.

Thus when one is faced with the task of coding an α-r.e. set X into an α-axiomatizable theory $T(X)$ of the same α-degree, it appears that for the elimination of quantifiers to succeed smoothly in the absence of general distributive laws, the Lindenbaum α-algebra of the statements of $T(X)$ must be atomic. This does not mean that an atomless part, even if suitably disguised, makes the problem hopeless.

The proof of Theorem 3.3 uses the theory of a linear ordering together with axioms saying that, at every ordinal stage β ($\beta < \alpha$), and after every two elements with only a finite number of elements between them are identified, the resulting linear ordering is either the ordering of one element or else every element has an immediate successor and an immediate predecessor (see [3]). The Lindenbaum α-algebra of statements of the theory is atomic. Every statement is an α-finite disjunction of atoms and negations of atoms. The set of atoms is {the ordering becomes the ordering of one element at stage $\beta \mid \beta < \alpha$}. Furthermore, any two models, where the ordering becomes the ordering of one element at a stage beyond α, are indistinguishable.

It should be remarked that, while the general distributive law fails in α-logic, properly restricted distributive laws can and are used in the proof of 3.3. Justification of the induction in the elimination of quantifiers involves a direct application of the generalized recursion theorem and is omitted.

Let T be the α-theory based on the axioms for a simple ordering on α with $<$ as the only primitive relation symbol, besides equality.

3.1. Definition.

$$
\begin{array}{ll}
x <_0 y & \text{iff } x < y; \\
x <_{\beta+1} y & \text{iff } \bigwedge_{n \in \omega} \exists v_0 <_\beta \cdots <_\beta v_n \wedge [(v_0 = x) \wedge (v_n = y)]; \\
x <_\lambda y & \text{iff } \bigwedge_{\beta \in \lambda} x <_\beta y, \ \lambda \text{ a limit ordinal}; \\
x \equiv_0 y & \text{iff } x = y; \\
x \equiv_\beta y & \text{iff } \neg(x <_\beta y) \wedge \neg(y <_\beta x).
\end{array}
$$

Intuitively, $x \equiv_{\beta+1} y$ means that, for some $\gamma \leq \beta$, factoring out according to $\equiv_\gamma$ resulted in x being a finite $<_{\gamma+1}$ predecessor of y. Let

$$
\begin{aligned}
T_\beta \equiv\ & [\forall x \forall y (x \equiv_\beta y)] \\
& \vee [\forall x \exists y \exists z \forall w [(y <_\beta x <_\beta z) \wedge (w <_\beta x \rightarrow w \leq_\beta y) \wedge (x <_\beta w \rightarrow z \leq_\beta w)]]
\end{aligned}
$$

and

$$V_\beta \equiv \forall x \forall y (x \equiv_{\beta+1} y) \wedge \exists x \exists y \neg (x \equiv_\beta y).$$

T_β is the axiom asserting that either $<_\beta$ is the ordering of a singleton or every

[3] Allowing α-finite Boolean meets and joins.

element has an immediate successor and an immediate predecessor under $<_\beta$. V_β is the statement saying that the ordering collapses to one element at stage $\beta + 1$ but not earlier.

Now let $T' = T \cup \{T_\beta \mid \beta < \alpha\}$ closed under α-deduction. For given fixed n and $\delta < \alpha$, let

$$\tau_n = [n(n-1)/2] + 1;$$

$$f\colon \tau_n + 1 \to \{0, 1\};$$

$$\psi^{f(k)} \equiv \psi \quad \text{if } f(k) = 0, \qquad k \le \tau_n + 1;$$
$$\equiv \delta = \delta \quad \text{if } f(k) = 1,$$

$$\varphi_\gamma \equiv V_\gamma \quad \text{if } \gamma < \delta,$$
$$\equiv \neg \bigvee_{\beta<\delta} V_\beta \quad \text{if } \gamma = \delta;$$

$$\varphi_{\gamma,k,i,j} \equiv I_{\gamma k}(x_i, x_j) \quad \text{if } \gamma < \delta,$$
$$\equiv \neg \bigvee_{n\in\omega;\,\beta<\delta} I_{\beta,n}(x_i, x_j) \quad \text{if } \gamma = \delta,$$

where $I_{\gamma,k}(x_i, x_j) \equiv$ "x_j is the kth successor of x_i under $<_\gamma$."

3.2. Lemma. *Given a formula of n free variables $\psi(x_0, \cdots, x_{n-1})$, there is a corresponding ordinal $\delta < \alpha$ and a T'-equivalent formula ψ^* which can be obtained α-effectively from ψ and is of the form*

(1) $$\bigvee_A \varphi^{f(\tau_n)} \wedge \varphi_0^{f(0)} \wedge \cdots \wedge \varphi_{\tau_n-1}^{f(\tau_n-1)},$$

where φ is a φ_γ for some $\gamma \le \delta$, each φ_l is a $\varphi_{\gamma,k,i,j}$ for some $\gamma \le \delta$, $k \le \tau_n$, $i < n$, $i < j < n$, and A is an α-finite subset of

$$B = [\alpha \cup \{\alpha\}] \times [(\alpha \cup \{\alpha\}) \times \omega]^{\tau_n} \times [\{0, 1\}^{\tau_n+1}].^4$$

Proof. (i) It is immediate that atomic formulas are in the form of (1).

(ii) Given an α-finite set of formulas $\{\psi_i \mid i \in M\}$, all in the form of (1) with corresponding ordinals $\{\delta_i \mid i \in M\}$ and index sets $\{A_i \mid i \in M\}$, let ψ_i' be the formula obtained from ψ_i by substituting $\delta = \text{lub}\,\{\delta_i \mid i \in M\}$ for δ_i in the formulation rules for the components of ψ_i. Then

$$T' \vdash \bigvee_{i\in M} \psi_i \leftrightarrow \bigvee_{j \in \bigcup A_i;\, i\in M} \psi_j'.$$

(iii) Showing closure under negation involves some slightly more elaborate manipulation. Given a formula of n free variables,

(2) $$\psi \equiv \bigvee_A \varphi^{f(\tau_n)} \wedge \varphi_0^{f(0)} \wedge \cdots \wedge \varphi_{\tau_n-1}^{f(\tau_n-1)}.$$

Define an equivalence relation $\sim_f$ on B by $z_1 \sim_f z_2$ iff the $(\tau_n + 2)$-tuples z_1 and z_2 have the same last entry. Let $B_1, B_2, \cdots, B_L$ be the partition of B in its $\sim_f$-equivalence classes, and let $A_i = B_i \cap A$, $i \le L$. Rewrite (2) as

$$\psi \equiv \bigvee_{A_1} \varphi^{f_1(\tau_n)} \wedge \cdots \wedge \varphi_{\tau_n-1}^{f_1(\tau_n-1)} \vee \cdots \vee \bigvee_{A_L} \varphi^{f_L(\tau_n)} \wedge \cdots \wedge \varphi_{\tau_n-1}^{f_L(\tau_n-1)}.$$

Then $\neg\psi \equiv \neg(\bigvee_{A_1} \cdots) \wedge \cdots \wedge \neg(\bigvee_{A_L} \cdots)$.

If each conjunct $\neg(\bigvee_{A_i} \cdots)$ can be put in the form of (1) then so can $\neg\psi$ by

---

[4] A typical member of A, $\langle \gamma, \langle \gamma_0, k_0\rangle, \cdots, \langle \gamma_{\tau_n-1}, k_{\tau_n-1}\rangle, f\rangle$ would yield the disjunct $\varphi_\gamma^{f(\tau_n)} \wedge \bigwedge_{l<\tau_n;\, n>i<j<n} \varphi_{\gamma_l,k_l,ij}^{f(l)}$.

distributing over the *finitely* many conjunctions and collecting terms. But we have that

$$T' \vdash \neg\Big(\bigvee_{A_i} \varphi^{f_i(\tau_n)} \wedge \cdots \wedge \varphi_{\tau_n - 1}^{f_i(\tau_n - 1)}\Big) \leftrightarrow \bigvee_{B_i - A_i} \varphi^{f_i(\tau_n)} \wedge \cdots \wedge \varphi_{\tau_n - 1}^{f_i(\tau_n - 1)},$$

since the disjuncts $\varphi^{f_i(\tau_n)} \wedge \cdots \wedge \varphi_{\tau_n-1}^{f_i(\tau_n-1)}$ are mutually exclusive. Noting that $B_i - A_i$ is α-finite, closure under negation is established.

(iv) To show that the existential quantifier can be eliminated consider the following cases for

$$\exists x_k \psi \equiv \exists x_k \bigvee_A \varphi^{f(\tau_n)} \wedge \cdots \wedge \varphi_{\tau_n - 1}^{f(\tau_n - 1)},$$
$$\equiv \bigvee_A \exists x_k(\varphi^{f(\tau_n)} \wedge \cdots \wedge \varphi_{\tau_n - 1}^{f(\tau_n - 1)}), \qquad k \leq n - 1,$$

δ is the corresponding ordinal.

Case I. x_k occurs freely in exactly one φ_i, $f(i) = 0$, $i \leq \tau_n - 1$.

(a) $\varphi^{f(\tau_n)} \equiv \varphi_\beta$ for $\beta < \delta$.

Without loss of generality suppose that $\varphi_i \equiv \varphi_{\gamma,m,k,j}$, $m \in \omega$, $j > k$. Then either $\gamma = \delta$ in which case

$$T' \vdash \neg(\exists x_0, \cdots, \exists x_{n-1}(\varphi^{f(\tau_n)} \wedge \cdots \wedge \varphi_{\tau_n - 1}^{f(\tau_n - 1)})),$$

because $\varphi_\beta \wedge \exists x_k(\neg \bigvee_{m \in \omega; \gamma < \delta} I_{\gamma,m}(x_k, x_j))$ is inconsistent; or $\gamma < \delta$ in which case

$$T' \vdash \exists x_k(\varphi^{f(\tau_n)} \wedge \cdots \wedge \varphi_{\tau_n - 1}^{f(\tau_n - 1)})$$
$$\leftrightarrow \varphi^{f(\tau_n)} \wedge \cdots \wedge \varphi_{i-1}^{f(i-1)} \wedge I_{0,0}(x_j, x_j) \wedge \varphi_{i+1}^{f(i+1)} \wedge \cdots \wedge \varphi_{\tau_n - 1}^{f(\tau_n - 1)}.$$

(b) $\varphi^{f(\tau_n)} \equiv \neg \bigvee_{\beta < \delta} V_\beta$ or $f(\tau_n) = 1$.

Then

$$T' \vdash \exists x_k(\varphi^{f(\tau_n)} \wedge \cdots \wedge \varphi_{\tau_n - 1}^{f(\tau_n - 1)})$$
$$\leftrightarrow \varphi^{f(n)} \wedge \cdots \wedge \varphi_{i-1}^{f(i-1)} \wedge I_{0,0}(x_j, x_j) \wedge \varphi_{i+1}^{f(i+1)} \wedge \cdots \wedge \varphi_{\tau_n - 1}^{f(\tau_n - 1)}.$$

Case II. x_k occurs freely in more than one φ_i with $f(i) = 0$. Then, for every pair of occurrences i, j with, say, $\varphi_i \equiv \varphi_{\gamma_i,m_i,k,l_i}$, $\varphi_j \equiv \varphi_{\gamma_j,m_j,l_j,k}$, consider two subcases:

(a) $\gamma_i > \delta$ and $\gamma_j > \delta$.

(i) If $\gamma_i = \gamma_j = \gamma$, then

$$T' \vdash \exists x_k(\varphi^{f(\tau_n)} \wedge \cdots \wedge \varphi_{\tau_n - 1}^{f(\tau_n - 1)})$$
$$\leftrightarrow (\varphi^{f(\tau_n)} \wedge \cdots \wedge \varphi_{i-1}^{f(i-1)} \wedge \varphi_{i+1}^{f(i+1)} \wedge \cdots \wedge \varphi_{j-1}^{f(j-1)}$$
$$\wedge \varphi_{\gamma,m,l_j,l_i} \wedge \varphi_{j+1}^{f(j+1)} \wedge \cdots \wedge \varphi_{\tau_n - 1}^{f(\tau_n - 1)}),$$

where $m = m_i + m_j$.

(ii) If $\gamma_i < \gamma_j$, then

$$T' \vdash \exists x_k(\varphi^{f(\tau_n)} \wedge \cdots \wedge \varphi_{\tau_n - 1}^{f(\tau_n - 1)})$$
$$\leftrightarrow (\varphi^{f(\tau_n)} \wedge \cdots \wedge \varphi_{i-1}^{f(i-1)} \wedge \varphi_{i+1}^{f(i+1)} \wedge \cdots \wedge \varphi_{j-1}^{f(j-1)}$$
$$\wedge \varphi_{\gamma_j,m_j,l_j,l_i} \wedge \cdots \wedge \varphi_{\tau_n - 1}^{f(\tau_n - 1)}).$$

(iii) If $\gamma_i > \gamma_j$, then

$$T' \vdash \exists x_k(\varphi^{f(\tau_n)} \wedge \cdots \wedge \varphi_{\tau_n - 1}^{f(\tau_n - 1)})$$
$$\leftrightarrow (\varphi^{f(\tau_n)} \wedge \cdots \wedge \varphi_{i-1}^{f(i-1)} \wedge \varphi_{i+1}^{f(i+1)} \wedge \cdots \wedge \varphi_{j-1}^{f(j-1)}$$
$$\wedge \varphi_{\gamma_i,m_i,l_j,l_i} \wedge \varphi_{j+1}^{f(j+1)} \wedge \cdots \wedge \varphi_{\tau_n - 1}^{f(\tau_n - 1)})$$

(b) If $(\gamma_i = \delta) \vee (\gamma_j = \delta)$, then

$$T' \vdash \exists x_k(\varphi^{f(\tau_n)} \wedge \cdots \wedge \varphi_{\tau_n-1}^{f(\tau_n-1)})$$
$$\leftrightarrow \Big(\varphi^{f(\tau_n)} \wedge \cdots \wedge \varphi_{i-1}^{f(i-1)} \wedge \varphi_{i+1}^{f(i+1)} \wedge \cdots \wedge \varphi_{j-1}^{f(j-1)}$$
$$\wedge \Big(\neg \bigvee_{\gamma<\delta;\, m\in\omega} I_{\gamma,m}(x_{l_j}, x_{l_i})\Big) \wedge \cdots \wedge \varphi_{\tau_n-1}^{f(\tau_n-1)}\Big).$$

Iterating the above procedure for every pair i, j for which $x_{l_j} < x_k < x_{l_i}$, the existential quantifier is eliminated.

3.3. THEOREM. *Let α be an admissible ordinal. Given an α-r.e. subset X of α, there is an α-theory $T(X)$ with an α-r.e. set of axioms, whose α-degree of unsolvability is the same as that of X.*

PROOF. Let $T(X) = T \cup \{\neg V_\beta \mid \beta \in X\}$ closed under α-deduction.

It is immediate that α-degree$(X) \leq \alpha$-degree$(T(X))$. To establish α-reducibility in the opposite direction, consider a given α-finite conjunction. By Lemma 3.2, we can α-effectively find the canonical form of σ which is

$$\sigma^* \equiv \bigvee_{i\in A} (V_{\beta_i} \vee \neg \bigvee_{\gamma \in M_i} V_\gamma),$$

where A and M_i's are α-finite.

Then $T(X) \vdash \sigma^*$ iff $\exists i \in A(M_i \neq \varnothing$ and $M_i \subset X)$. Thus α-degree$(X) = \alpha$-degree$(T(X))$.

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